1.5b first-order linear systems

Monday, January 18, 2021 2:12 AM

Recall: Given a kth-order ODE $f(x^{(k)}, x^{(k-1)}, \dots, \dot{x}, x, t) = 0$, we can convert it to a system of k 1st-order ODEs.

Ex. $\ddot{x} + 2\dot{x} - x = t^2 \longrightarrow$

$$\left.\begin{array}{l} y_1 = x \\ y_2 = \dot{x} = \dot{y}_1 \\ \dot{y}_2 = \ddot{x} = \ddot{y}_1 \end{array}\right\} \quad \begin{array}{l} y_2 = \dot{y}_1 \\ \dot{y}_2 + y_2 - y_1 = 0 \end{array}$$

Let $x(t+k) + a_{k-1}x(t+k-1) + \dots + a_1 x(t+1) + a_0 x(t) = b(t)$

Let

$$Y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ \vdots \\ y_k(t) \end{bmatrix}$$

and

$$\left.\begin{array}{l} y_1(t) = x(t) \\ y_2(t) = x(t+1) = y_1(t+1) \\ y_3(t) = x(t+2) = y_2(t+1) \\ \vdots \\ y_k(t) = x(t+k-1) = y_{k-1}(t+1) \\ \text{Then } y_k(t+1) + a_{k-1}y_k(t) + \dots + a_1 y_2(t) + a_0 y_1(t) = b(t) \end{array}\right\} \begin{array}{l} k \text{ 1st-order} \\ \text{difference} \\ \text{equations} \end{array}$$

Or equivalently, $Y(t+1) = AY(t) + B(t)$, where

$$A = \underbrace{\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{k-1} \end{pmatrix}}_{\text{companion matrix}} \quad \text{and} \quad B(t) = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ b(t) \end{pmatrix}$$

Given a system $X(t+1) = AX(t) + B(t)$, use Principle of Superposition.

If $X_h(t+1) = AX_h(t)$ and $X_p(t+1) = AX_p(t) + B(t)$, then $X_h + X_p$ is a sol.

Also, $X_h(t) = \sum^{k} c_i X_i(t)$ where $X_i(t)$ are lin. ind. sol. to hom. equation.

Also, $X_h(t) = \sum_{i=1}^{k} c_i X_i(t)$, where $X_i(t)$ are lin. ind. sol. to hom. equation.

Sol. to hom. eq.: $X_h(t+1) = A X_h(t)$.

Ansatz: $X_h(t) = \lambda^t V$, $\lambda \in \mathbb{R}$, $V \in \mathbb{R}^k$

Then $\lambda^{t+1} V = \lambda^t V$

$\Rightarrow \lambda V = A V$

$\Rightarrow (\lambda, V)$ is an eigenpair of A.

If A has k distinct eigenvalues, then also k lin. ind. eigenvectors, so the sol of the form $\lambda_i^t V_i$ are linearly ind.

Sometimes, will have $t^n \lambda^t V$ if not k distinct eigenvalues.

Note, if eigenvalues are complex, often we will want to write in real form, so we get things like $t^n r^t \sin(\phi t) V$.

Def. 1.9 If $A \in \mathbb{R}^{k \times k}$ has k eigenvalues $\lambda_1, \dots, \lambda_k$, then the spectral radius $\rho(A) = \max_{i \in \{1,\dots,k\}} \{|\lambda_i|\}$.

Thm 1.1 Let $A \in \mathbb{R}^{k \times k}$. Then $\rho(A) < 1$ iff $\lim_{t \to \infty} A^t = 0$.

pf. Recall that any $A \in \mathbb{C}^{k \times k}$ is triangularizable to

$PAP^{-1} = T$, where T is upper triangular with eigenvalues along the diagonal and P is an invertible matrix.

Then $A^t = P^{-1}[PAP^{-1}]^t P = P^{-1} T^t P$.

$\lim_{t \to \infty} T^t = 0$ because the diagonal goes to 0 iff $|\lambda_i| < 1$. $\square$

$\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda = 1$, mult. 2

$\Rightarrow 1, t$ are lin. ind. sol.

Ex. 1.11 $x(t+2) - 2x(t+1) + x(t) = \cos t$

$\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda = 1$, mult. 2

$\Rightarrow 1, t$ are lin. ind. sol. to homog. eq.

$x(t)$

Ex. 1.11 $\quad x(t+2) - 2x(t+1) + x(t) = \cos t$

let $\quad y_1(t) = x(t)$

$y_2(t) = x(t+1) = y_1(t+1)$

$y_2(t+1) - 2y_2(t) + y_1(t) = \cos t$

$$\left.\begin{aligned} y_1(t+1) &= 0\,y_1(t) + y_2(t) \\ y_2(t+1) &= -y_1(t) + 2y_2(t) + \cos t \end{aligned}\right\}$$

Matrix form $\quad Y(t+1) = AY(t) + B(t)$ where

$$Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ \cos t \end{bmatrix}$$

Hom. eq. $\quad Y_h(t+1) = AY_h(t)$.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda - 2 \end{vmatrix} = \lambda(\lambda-2) + 1 = \lambda^2 - 2y + 1 = 0$$

$\Rightarrow (\lambda - 1)^2 = 0$

$\lambda = 1$

$$\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} y_2 &= y_1 \\ -y_1 + 2y_2 &= y_2 \end{aligned}\right\} \quad y_1 = y_2 \quad \text{eigenvector is } \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\Rightarrow \lambda^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{one solution}$$

$\leftarrow 1$

Another solution using methods from last time: $\begin{bmatrix} t \\ t+1 \end{bmatrix}$

$$\begin{bmatrix} t+1 \\ t+2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} t \\ t+1 \end{bmatrix} = \begin{bmatrix} t+1 \\ -t+2t+2 \end{bmatrix} = \begin{bmatrix} t+1 \\ t+2 \end{bmatrix}$$

$$\begin{bmatrix} t+1 \\ t+2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} t \\ t+1 \end{bmatrix} = \begin{bmatrix} t+1 \\ -t+2t+2 \end{bmatrix} = \begin{bmatrix} t+1 \\ t+2 \end{bmatrix}$$

$$Y_h(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} t+1 \\ t+2 \end{bmatrix} \quad \text{for some } c_1, c_2 \in \mathbb{R}.$$

Ex 1.13. $X(t+1) = A\,X(t)$

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 3 \\ 2 & \lambda - 1 \end{vmatrix} = (\lambda - 2)(\lambda - 1) - 6$$

$$= \lambda^2 - 3\lambda + 2 - 6$$

$$= \lambda^2 - 3\lambda - 4$$

$$= (\lambda - 4)(\lambda + 1) = 0$$

$$\lambda = -1, 4.$$

$$\lambda_1 = -1 \qquad \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 2x_1 + 3x_2 \\ 2x_1 + x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} \Rightarrow \begin{aligned} 2x_1 + 3x_2 &= -x_1 \\ 3x_1 &= -3x_2 \\ x_1 &= -x_2 \end{aligned}$$

$$\lambda_2 = 4 \qquad \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 2x_1 + 3x_2 \\ 2x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \end{bmatrix} \Rightarrow \begin{aligned} 2x_1 + 3x_2 &= 4x_1 \\ 3x_2 &= 2x_1 \end{aligned}$$

Sol. $$X(t) = c_1 (-1)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 (4)^t \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Sol. $X(t) = c_1 (-1)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 (4)^t \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} c_1 (-1)^t + 3 c_2 (4)^t \\ c_1 (-1)^{t+1} + 2 c_2 (4)^t \end{bmatrix}.$$

Also, $\rho(A) = 4$

$\Rightarrow \lim_{t \to \infty} A^t \neq 0.$